Symbolic Analysis of Discrete-Time Polynomial Systems (Theory and Application in Computer Algebra)

Anai, Hirokazu

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Symbolic Analysis of Discrete-Time Polynomial Systems

穴井 宏和 (Hirokazu ANAI) *

High Performance Computing Research Center,

FUJITSU LABORATORIES LTD.

4-1-1 Kamikodanaka, Nakahara-ku, Kawasaki 211-8588, JAPAN

TEL: +81-44-754-2670 / FAX: +81-44-754-2666

Abstract

In this paper we focus on the analysis of discrete-time polynomial systems. We present the modified algorithms for testing observability and accessibility based on the methods proposed in [6]. Our method not only quantifier elimination but also utilize Gröbner basis and real root counting techniques for decision problem computation too and incorporates the strategy to improve efficiency in checking equivalence of ideals.

Then we also have shown that the invertibility check of the polynomial systems can be achieved by using Gröbner basis technique. And the procedure for invertibility check yields the inverse if one exists at the same time. So backward accessibility and transitivity can be tested in the analogous way of our algorithm for forward accessibility check in finite time step.

1 Introduction

It is important for any in-depth analysis of the systems to understand the fundamental properties of the systems such as observability or controllability *etc.* In [6], D.Nešić presented the two algorithms arising in analysis of discrete-time polynomial systems. One is the algorithm for checking *"observability (OB)"* and the other is that for checking *"forward accessibility (FA)"*.

As for observability test, D.Nešić used the same scheme used in Sontang's work [9] and presented actually constructive symbolic algorithm, which always stops in finite time, by utilizing computer algebra methods such as *Gröbner Basis (GB)* and *quantifier elimination (QE)*.

As for forward accessibility test, there is the work by B.Jakubczyk *et.al.* [5] in 1990, where the Lie algebra methods were used to test accessibility. Here we take another

*e-mail:anai@flab.fujitsu.co.jp

approach used in [6] which is applicable to only polynomial systems. D.Nešić presented the constructive symbolic algorithm for checking forward accessibility, which is based on ideas of invariant algebraic set, by Gröbner basis and quantifier elimination .

However, there is some room for the method proposed by D.Nešić to make an improvement in view of the computational complexity. And if we have the inverse of the polynomial maps, we can actually realize the method to check "*backward accessibility (BA)*" and "*transitivity (TR)*" in the same way as FA check. In this paper we present the improved (algebraic) algorithms for checking OB and FA and extend the applicability of our FA check algorithms for checking to BA and TR check by introducing concrete procedure to compute inverse of polynomial systems by using GB.

2 Preliminary

We denote the ring of polynomials in x over a field K by $K[x]$. We use $\mathbf{R}$ and $\mathbf{Q}$ for fields of real numbers and rational numbers respectively and denote natural numbers by $\mathbf{N}$.

In this paper we consider the following class of polynomial systems:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \tag{1}$$

where $x(k) \in \mathbf{R}^n$, $y(k) \in \mathbf{R}$ and $u(k) \in \mathbf{R}$ are the state, output and input of the system (1) at time k respectively. Let $f(x,u) = (f_1(x,u), \cdots, f_n(x,u))$ and assume that $f_i(x,u) \in \mathbf{Q}[x,u]$ and $h \in \mathbf{Q}[x]$ (due to the applicability of symbolic computation packages). Note that, in general, f is a nonlinear.

We mostly will follow the notations in [6]. We denote the composition of f by

$$f_{u(1)} \circ f_{u(0)} = f(f(x(0), u(0)), u(1))$$

Let a sequence of controls (*i.e.* inputs) be $U = (u(0), u(1), \cdots)$ and its truncation of U to a sequence of length s is denoted as $U_s = (u(0), u(1), \cdots, u(s-1))$. We denote by $y(s, x(0), U_s)$ the output of system (1) which is reached from the initial state $x(0)$ at time step s under the action of a control sequence U_s.

Definition 1 (Observability) *The system (1) is observable if for each pair of initial states ξ, η, there exists an integer N and an input sequence U_N which yields $y(N, \xi, U_N) \neq y(N, \eta, U_N)$.*

We employ the following definition of invertibility for polynomial maps. Here $\mathbf{Q}(u)$ is the set of all (rational) polynomials in u and called a rational function field in u.

Definition 2 *Let $X = \{x_1, \cdots, x_n\}$. We consider an n-tuple $f = (f_1(X), \cdots, f_n(X)) \in (\mathbf{Q}(u)[X])^n$. f gives rise to a map*

$$\begin{aligned} \varphi_f : \quad (\mathbf{Q}(u))^n &\longrightarrow (\mathbf{Q}(u))^n \\ (a_1, \cdots, a_n) &\longmapsto (f_1(a), \cdots, f_n(a)), \end{aligned}$$

where $a = (a_1, \cdots, a_n) \in (\mathbf{Q}(u))^n$. φ_f *is called* invertible *if there exit* $g_1, \cdots, g_m \in \mathbf{Q}(u)[X]$ *such that*

$$g_i(f_1, \cdots, f_n) = X_i \quad for \quad 1 \le i \le n$$

so that $\varphi_g \circ \varphi_f = id_{(\mathbf{Q}(u))^n}$ *where* $id_{(\mathbf{Q}(u))^n}$ *is an identity on* $(\mathbf{Q}(u))^n$. *We denote the inverse of* f *by* f^{-1}.

We denote by $R_k^+(x(0))$ the set of states attainable from $x(0)$ in k forward steps, and by $R^+(x(0))$ the set of states attainable from $x(0)$ in any nonnegative number of forward steps. Replacing forward steps by backward steps we similarly obtain $R_k^-(x(0))$ and $R^-(x(0))$, which consists of states controllable to $x(0)$ in k steps, and controllable to $x(0)$ in any nonnegative number of steps, respectively. We denote the set of reachable states from an initial state $x(0)$ at step k by

$$V_r^k(x(0)) = \{x | x = f_{u(k-1)} \circ \cdots \circ f_{u(0)}(x(0)),\ u(i) \in \mathbf{R}\}$$

we can rewrite $R_k^+(x(0)) = \{z \mid z \in \bigcup_i^k V_r^i(x(0))\}$. The set of states attainable from $x(0)$ in any number of positive and negative steps is called the *orbit* of $x(0)$ and is denoted by $R(x(0))$.

Definition 3 (Accessibility) *The system (1) is* forward accessible from *the* $x(0) \in \mathbf{R}^n$ *if* $R^+(x(0))$ *has a non-empty interior (i.e.* $dim R^+(x(0)) = n$*).*

When the system (1) is invertible, we can define followings: the system (1) is backward accessible from $x(0) \in \mathbf{R}^n$ *if* $R^-(x(0))$ *has a non-empty interior. It is called* transitive from $x(0)$ *(or forward-backward accessible) if its orbit* $R(x(0))$ *has a non empty interior.*

Finally, the system is forward (backward) accessible *if it is forward (backward) accessible from any states* $x \in \mathbf{R}^n$*, and it is called* transitive *if it is transitive from any* $x \in \mathbf{R}^n$.

There is a following straightforward criterion for accessibility of the discrete time system (in [5]), which is based on the rank of Jacobian of a map $F_{k-1} = f_{u(k-1)} \circ \cdots \circ f_{u(0)}$. From the proposition shown in [5], we have

Proposition 4 *For fixed* x *and* k*, the interior of attainable set* $R_k^+(x)$ *from* x *is nonempty iff there exists an input sequence* $\mathbf{u}_{k-1} = (u_0, \cdots, u_{k-1})^T$ *such that*

$$rank\ \frac{\partial F_{k-1}}{\partial \mathbf{u}_{k-1}} = n.$$

The system (1) is forward accessible from x *iff there exists an* $\mathbf{u}_{k-1} = (u_0, \cdots, u_{k-1})^T$ *such that*

$$rank\ \frac{\partial F_{k-1}}{\partial \mathbf{u}_{k-1}} = n \quad for\ k \ge 1.$$

Hereafter, we consider the case of SISO(single input single output) system (1). The algorithm for SISO case is also directly applicable to MIMO (multi input multi output) case.

3 Checking observability

In this section we review the algorithm for checking observability proposed in [6]. We consider all states $\xi, \eta \in \mathbf{R}^n$ (such that $\xi \neq \eta$) which generate the same output sequence independently of the applied input sequence. Here we call such states *"indistinguishable"* states.

Framework: We construct a set which involves all the indistinguishable states; If two distinct states $\xi, \eta \in \mathbf{R}^n$ can not be distinguished by any input sequence, as a necessary condition, we have

$$h(\xi) = h(\eta) \tag{2}$$

at first time step. If the two states ξ, η generate the same output up to second time step independently of the applied input sequence, we necessarily have

$$h(f(\xi, u)) = h(f(\eta, u)) \tag{3}$$

in addition to (2). If we denote

$$h(f(x,u)) = h_m(x)u^m + \cdots + h_1(x)u + h_0(x),$$

then the condition (3) is equivalent to

$$h_i(\xi) = h_i(\eta) \ for \ \forall i = 0, 1, \cdots, m. \tag{4}$$

Then, the states ξ, η which have the same output for the first three steps satisfy

$$h(f \circ f(\xi, u)) = h(f \circ f(\eta, u)). \tag{5}$$

Let

$$h_i(f(x,u)) = h_{p_i,i}(x)u^{p_i} + \cdots + h_{1,i}(x)u + h_{0,i}(x).$$

Then the condition (5) is equivalent to

$$h_{p_i,i}(\xi) = h_{p_i,i}(\eta) \ for \ \forall i = 0, 1, \cdots, m. \tag{6}$$

We can obtain the necessary condition of indistinguishable states for any time step by continuing the same procedure as above.

We consider the sequence of ideals corresponding to that of necessary condition of indistinguishable states as follows;

$$J_1 = \langle h(\xi) - h(\eta) \rangle$$

$$J_2 = \langle h(\xi) - h(\eta), h_0(\xi) - h_0(\eta), \cdots, h_1(\xi) - h_1(\eta) \rangle$$

$$J_3 = \langle h(\xi) - h(\eta), h_0(\xi) - h_0(\eta), \cdots, h_1(\xi) - h_1(\eta),$$
$$h_{0,0}(\xi) - h_{0,0}(\eta), \cdots, h_{p_m,m}(\xi) - h_{p_m,m}(\eta) \rangle$$

$$\vdots$$

Then by the construction we have the ascending chain of ideals

$$J_1 \subset J_2 \subset J_3 \subset \cdots . \tag{7}$$

The ascending chain (7) must terminate in finite length *i.e.* there exists an integer N such that $J_N = J_{N+1}$ (by Noetherian property of the polynomial ring). Let the variety of the ideal J_N be $V(J_N)$. We consider the set

$$S_z = V(J_N) \ \cap \ \{(\xi, \eta) \mid \xi \neq \eta\}.$$

Then all the indistinguishable states belong to the set S_z. Let the set of equations obtained by setting all generators of the ideal J_N equal to zero be E_{J_N} . Next we check the decision problem:

$$\exists \xi \ \exists \eta \ (E_{J_N} \ \wedge \ \xi \neq \eta) \tag{8}$$

If the answer to (8) is *false* ($S_z = \emptyset$), the system is observable, otherwise (answer is *true*; $S_z \neq \emptyset$) the system is not observable.

Realization: Here we need two symbolic method, that is, GB and QE. Testing the coincidence of the ideals J_i and J_{i+1} for $\forall i$ is achieved by comparing their reduced GB with respect to the same ordering. And we can determine the decision problem (8) by using QE since S_z is obviously a semi-algebraic set in $\mathbf{R}^n \times \mathbf{R}^n$.

4 Checking forward accessibility

In this section we review the algorithm for checking forward accessibility proposed in [6].

Framework: First consider the composition;

$$F = f_{u_{n-1}} \circ \cdots \circ f_{u_0}$$

and its Jacobian:

$$J = \frac{\partial F}{\partial \mathbf{u}_{n-1}}.$$

where $\mathbf{u} = (u_0, \cdots, u_{n-1})^T$. Let the determinant of J be

$$det\ J = \sum_{i_0=0}^{N_0} \cdots \sum_{i_{n-1}=0}^{N_{n-1}} b_{i_0, i_1, \cdots, i_{n-1}}(x)\ u_0^{i_0} \cdots u_{n-1}^{i_{n-1}}.$$

and $\mathcal{J}_C = \langle b_{0,\cdots,0}, \cdots, b_{N_0,\cdots,N_{n-1}} \rangle$. We define *critical variety* V_C by $V_C = V(\mathcal{J}_C)$.

From proposition 4, if there exists a state x such that the determinant of J is non-zero polynomial in $\mathbf{u}$, then the system (1) is forward accessible from x. So the critical variety V_C contains all states $x \in \mathbf{R}^n$ which satisfy $dimV_r^n(x) < n$. Here we define the key notion "invariant set" in this algorithm.

Definition 5 *$S_I \subset V_C$ is an* invariant set *if $f(x(0)\ ,u) \in S_I$ for all $x(0) \in S_I$ and $u \in \mathbf{R}$. $V_I \subset V_C$ is called* maximal invariant set *if it satisfies that if $V_I \subset V^*$, where $V^* \in V_C$ is an invariant set, then $V_I = V^*$.*

Theorem 6 (Nešić [6]) *Consider a system (1) and assume that* det $J \neq 0$. *Then the system is forward accessible if and only if $V_I = \emptyset$.*

Now we mention the computation of V_I. Consider the ideal $\mathcal{J}_C$. We use the notation

$$\mathcal{J}_C \circ f_u(x) = \mathcal{J}_C(f(x,u)).$$

And let $\mathcal{J}_C = \langle g_1, \cdots, g_s \rangle$ then we have

$$\begin{aligned} \mathcal{J}_C \circ f_u(x) &= \langle g_1(f(x,u)), \cdots, g_s(f(x,u)) \rangle. \\ &= \langle g_1 \circ f_u(x), \cdots, g_s \circ f_u(x) \rangle. \end{aligned}$$

where note that $g_j \circ f_u(x) = g_j(f(x,u)) \in \mathbf{Q}[x_1, \cdots, x_n][u]$. So let

$$g_j \circ f_u(x) = \sum_{i=0}^{N_j} a_{j,i}(x) u^i, \quad j = 1, 2, \cdots, s.$$

And consider the ideal

$$\mathcal{J}_C^{(1)} = \mathcal{G}_1 \cdot \mathcal{G}_2 \cdot \cdots \cdot \mathcal{G}_s$$

where $\mathcal{G}_j = \langle a_{j,0}, a_{j,1}, \cdots, a_{j,N_j} \rangle$. (The products of two ideals $\mathcal{I}_1 = \langle a_1, \cdots, a_N \rangle$ and $\mathcal{I}_2 = \langle b_1, \cdots, b_M \rangle$ is defined by $\mathcal{I}_1 \cdot \mathcal{I}_2 = \langle a_i b_j | 1 \leq i \leq N, 1 \leq j \leq M \rangle$. The sum of two ideals $\mathcal{I}_1, \mathcal{I}_2$ is defined by $\mathcal{I}_1 + \mathcal{I}_2 = \langle a_1, \cdots, a_N, b_1, \cdots, b_M \rangle$.) We can form the ideal $\mathcal{J}_C^{(k)}$ in the same way by taking composition

$$\mathcal{J}_C \circ f_{u(k-1)} \circ \cdots \circ f_{u(0)(x)},$$

regarding its generators as polynomials in $u(i)$, $i = 0, 1, \cdots, k-1$, with coefficients polynomials in x and achieving the same manner as before.

Let $\mathcal{J}_0 = \mathcal{J}_C$ and $\mathcal{J}_t = \sum_{k=0}^{t} \mathcal{J}_C^{(k)}$ for $t = 0, 1, 2, \cdots$. Continuing the same procedure we have an ascending chain of ideals

$$\mathcal{J}_0 \subset \mathcal{J}_1 \subset \mathcal{J}_2 \subset \cdots \tag{9}$$

The ascending chain (9) must terminate in finite length *i.e.* there exists an integer N such that $\mathcal{J}_N = \mathcal{J}_{N+1}$ (by Noetherian property of the polynomial ring). Let $\mathcal{J}_I = \sum_{k=0}^{N} \mathcal{J}_i$. Then the maximal invariant set V_I of V_C is given by $V(\mathcal{J}_I)$.

Let the set of equations obtained by setting all generators of the ideal $\mathcal{J}_I$ equal to zero be $E_{\mathcal{J}_I}$. Next we check the decision problem:

$$\exists x (E_{\mathcal{J}_I}) \tag{10}$$

If the answer to (10) is *false* ($V_I = \emptyset$), the system is forward accessible, otherwise (answer is *true*; $V_I \neq \emptyset$) the system is not forward accessible.

Realization: Here we need two symbolic method, that is, GB and QE. Testing the coincidence of the ideals $\mathcal{J}_i$ and $\mathcal{J}_{i+1}$ for $\forall i$ is achieved by comparing their reduced GB with respect to the same ordering. And we can determine the decision problem (10) by using QE.

5 Improvements of algorithms

Since GB and QE are guaranteed to terminate in finite time, two algorithms proposed in [6] terminate in finite time. But the computational complexity of the methods is not small. So, it is very important to improve the algorithm in view of the complexity when, in particular, we consider the industrial applications. In this section, we will present the improvements of the algorithms.

5.1 First improvement

In [6], as for OB check, the case where we do not have to use QE is indicated; If the obtain reduced GB of the ideal J_k at some step k is $\{\xi - \eta\}$, the system is observable since it implies that $J_k = J_{k+1}$ and $V(J_k) = V(\xi - \eta)$. This is a trivial case where we do not have to use QE.

Here we propose the algorithms for observability and forward accessibility test which does not use directly QE for decision problems. We use GB and real root counting method for multivariate polynomial systems before applying QE to them.

OB : The decision problem (8) can be checked by using GB as follow; We introduce new slack variable t and consider the ideal I_d

$$I_d = \langle J_N \cap (\xi - \eta)t - 1 \rangle. \tag{11}$$

Let the GB of I_d with respect to the block ordering $\{\xi, \eta\} \lll \{t\}$ be G_d.

If G_d is equal to $\{1\}$, ($S_z = \emptyset$), the system is observable. If G_d is not equal to $\{1\}$, G_d has some roots in $\mathbf{C}$. But in this case G_d may have no real roots, then the system is observable. So next we check whether G_d has a real root or not. Since we already have G_d, we can determine whether G_d is 0-dimentiomal or not immediately. If G_d is 0-dimensional, we count the number of real roots of G_d by using the method by P.Pedersen *et.al.*[8]. Then if the number of real roots is equal to 0, the system is observable. If G_d is not 0-dimensional, we use QE as proposed by D.Nešić.

This yields the more efficient algorithm than original one since in many case actually G_d is equal to $\{1\}$ by the construction and, in general, complexity of GB and real root counting is smaller than that of QE.

Remark 1 *In actual procedure, we construct the chain of Gröbner basis* G_i *of the* J_i $(i = 1, 2, \cdots)$*. Hence, in order to obtain the Gröbner basis* G_d *of* I_d*, we compute the Gröbner basis of the ideal* $I'_d = \langle G_N \cap (\xi - \eta)t - 1 \rangle$.

FA : The decision problem (10) can be checked by the same way as above; if the Gröbner basis G_I of $\mathcal{J}_I$ is $\{1\}$, the system (1) is forward accessible. If G_I is not equal to $\{1\}$, G_d has some roots in $\mathbf{C}$. But in this case G_I may has no real roots, then the system is observable. So next we check whether G_I is 0-dimentiomal or not. If G_I is 0-dimensional, we count the number of real roots of G_I similarly. If G_I is not 0-dimensional, we use QE. This leads to the improvement of efficiency by the same reasons as explained for FA check.

5.2 Second improvement

Here we mention the improvement for the efficiency of the computation of ascending chain and its equivalence check appearing two algorithms.

OB : We denote the set of polynomials which are added to J_{k-1} in order to construct J_k at step k by A_k for $k = 1, 2, \cdots$. (Here for convenience' sake let $J_0 = \langle\rangle$.);

$$A_1 = \{h(\xi) - h(\eta)\}$$

$$A_2 = \{h_0(\xi) - h_0(\eta), \cdots, h_1(\xi) - h_1(\eta)\}$$

$$A_3 = \{h_{0,0}(\xi) - h_{0,0}(\eta), \cdots, h_{p_m,m}(\xi) - h_{p_m,m}(\eta)\}$$

$$\vdots$$

Now we can rewrite

$$J_1 = \langle A_1 \rangle$$

$$J_2 = J_1 + \langle A_2 \rangle$$

$$J_3 = J_2 + \langle A_3 \rangle = \langle A_1 \rangle + \langle A_2 \rangle + \langle A_3 \rangle$$

$$\vdots$$

where for two ideals $\mathcal{I}_1 = \langle p_1, \cdots, p_s \rangle$ and $\mathcal{I}_2 = \langle q_1, \cdots, q_t \rangle$, the sum of two ideals stands for $\mathcal{I}_1 + \mathcal{I}_2 = \langle p_1, \cdots, p_s, q_1, \cdots, q_t \rangle$.

In the case of nonlinear systems (*i.e.* $deg(f) > 1$), the degree of polynomials in A_i which are adjoined to construct the next ideal of ascending chain J_i become large according to $deg(h)^i$. Here we propose the improvement using *normal form computation* in order to circumvent the degree explosion of polynomials in A_k $(k = 1, 2, \cdots)$ successively when we compute GB of J_{k+1}.

Consider *k-th* step. We have computed Gröbner basis G_{k-1} of J_{k-1}. Let $A_k = \{a_1, \cdots, a_s\}$. Now what we do is to compute Gröbner basis of J_k. First we compute the normal form of a_i with respect to G_{k-1} for all $i = 1, \cdots, s$. And we denote the set of the normal forms which is not equal to 0 by $A'_k = \{a'_1, \cdots, a'_{s'}\}$ where $s' \leq s$. Here if every normal form of a_i is 0 (*i.e.* $A'_k = \emptyset$), then we have $G_{k-1} = G_k$. Otherwise, consider the ideal

$$J'_k = G_{k-1} + \langle A'_k \rangle,$$

then we obviously have by the properties of Gröbner basis :

Corollary 7 *The Gröbner basis of* $J'_k = G_{k-1} + \langle A'_k \rangle$ *is equivalent to the Gröbner basis* G_k *of* $J_k = J_{k-1} + \langle A_k \rangle$.

By this corollary we can construct the Gröbner basis G_k by computing Gröbner basis of J'_k instead of J_k. This leads to the improvement of the efficiency owing to the following reasons; The computation of Gröbner basis of J'_k is much more efficient than that of J_k because the number and degree of polynomials of A'_k may be smaller (are mostly small) than those of A_k and we use G_k instead of J_k. Furthermore, in general, computational cost of normal form computation is much small compared to the complexity of Gröbner basis computation.

FA : The set of polynomials which are added to $\mathcal{J}_{k-1}$ in order to construct $\mathcal{J}_k$ at step k is $\mathcal{J}_C^{(k)}$ for $k = 1, 2, \cdots$. We can improve the procedure for checking equivalence of ideals $\mathcal{J}_i$'s in the same manner as the case OB check. This leads to the improvement of efficiency by the same reasons as explained above.

Here we show modified algorithms for OB, FA check based on Gröbner basis computation. Here we use abbreviation $GB[I]$ for GB of the idal I.

Algorithm 1[Modified OB check]
Input: a polynomial system $(f(x,u), h(x))$
output: observable or not

1. $k = 1$, fix a monomial ordering.
2. Let $J_1 = \langle h(\xi) - h(\eta) \rangle$. Compute $G_1 = GB[G_1]$.
3. $k = k + 1$.
4. Construct A_k and let $A_k = \{a_1, \cdots, a_{s(k)}\}$. Compute the normal form m_i of a_i with respect to G_{k-1} for $i = 1, 2, \cdots, s$.
   And let the set of all non-zero m_i be A'_k. If $A'_k = \emptyset$ then go to 5.
   Otherwise compute $G_k = GB[J_k]$ where $J'_k = G_{k-1} + \langle A'_k \rangle$ and go to 3.
5. Compute the Gröbner basis $G_d (\equiv G_{k-1})$ of the ideal I'_d. If $G_d = \{1\}$ then the system is observable.
6. If $G_d \neq \{1\}$ then check whether G_d is 0-dimentiomal or not by G_d.
   (a) If G_d is 0-dimensional, we count the number of real roots of G_d. Then if the number of real roots is 0, the system is observable.
   (b) If G_d is not 0-dimensional, we use QE for the decision problem (8). If (8) is false, the system is observable. Otherwise the system is not observable.

Algorithm 2[Modified FA check]
Input: a polynomial system $(f(x,u), h(x))$
output: forward accessible or not

1. $k = 0$, fix a monomial ordering.
2. Compute $G_0 = GB[\mathcal{J}_0]$.
3. $k = k + 1$.
4. Construct $\mathcal{J}_C^{(k)}$ and let $\mathcal{J}_C^{(k)} = \{g_1, \cdots, g_{s(k)}\}$. Compute the normal form m_i of g_i with respect to G_{k-1} for $i = 1, 2, \cdots, s$.
   And let the set of all non-zero m_i be $\mathcal{A}'_k$. If $\mathcal{A}'_k = \emptyset$ then go to 5.
   Otherwise compute $G_k = GB[\mathcal{J}'_k]$ where $\mathcal{J}'_k = G_{k-1} + \langle \mathcal{A}'_k \rangle$ and go to 3.
5. Compute the Gröbner basis $G_I (\equiv G_{k-1})$ of the ideal $E_{\mathcal{J}_I}$. If $G_I = \{1\}$ then the system is forward accessible,
6. If $G_I \neq \{1\}$ then check whether G_I is 0-dimentiomal or not by G_I.
   (a) If G_I is 0-dimensional, we count the number of real roots of G_I. Then if the number of real roots is 0, the system is observable.
   (b) If G_I is not 0-dimensional, we use QE for the decision problem (10). If (10) is false, the system is observable. Otherwise the system is not observable.

6 Invertibility and extension to BA and TR check

One of main difficulties in the studies of the general discrete-time case is due to the possible non-invertibility of the one-step transition maps $x \mapsto f(x,u)$. It is very important to propose the concrete procedure for determining the invertibility of the map and computing the inverse if one exists. Since we consider the discrete-time polynomial system here, we can propose the method to do it. If we can compute the inverse of systems, we can apply our algorithm of FA check to the BA and TR check.

Fortunately, we have following theorem for invertibility of polynomial maps [2]:

Proposition 8 *Let $f = (f_1, \cdots, f_n) \in (\mathbf{Q}(u)[X])^n$, $T = \{Y_1, \cdots, Y_n\}$ be a new indeterminate and*

$$I = \langle Y_1 - f_1, \cdots, Y_n - f_n \rangle.$$

Furthermore, let $\preceq$ be a term order on $T(X,Y)$ that satisfies $Y \lll X$. Then, φ_f is invertible if and only if the reduced Gröbner basis of I with respect to $\preceq$ is of the form

$$G = \{X_1 - g_1, \cdots, X_n - g_n\}$$

with $g_1, \cdots, g_n \in \mathbf{Q}(u)[Y]$. Moreover if (b) holds, then $g_i(f_1, \cdots, f_n) = X_i$ for $1 \geq i \geq n$.

By this proposition, for a given f, Gröbner basis (over a rational function field in u) is a method to decide whether φ_f is invertible or not and computing the inverse if exists at the same time.

Consider the case where f_u of the system (1) is invertible and assume the inverse f^{-1} of f_u has computed by using GB as shown above. The maps f_u and their inverses f_u^{-1} can be considered as "one step forward maps" and "one step backward maps" respectively. If we apply a sequence of controls $u_0, \cdots, u_{k-1}$ and allow backward as well as forward steps, we obtain a larger family of composition maps

$$\mathcal{F}_{k-1} = f_{u_{k-1}}^{e_{k-1}} \circ \cdots \circ f_{u_0}^{e_0}. \tag{12}$$

where each e_i takes a value ± 1.

Then, backward accessibility is checked by applying the same manner in §5,6 to $\mathcal{F}_{n-1}$ such that e_i are -1 for $i = 0, 1, \cdots, n-1$.

Transitivity check is also achieved by using analogous argument in §5,6 for the map $\mathcal{F}_{n-1}$ since the orbit is the countable union of the images of the map $\mathcal{F}_{k-1}$.

7 Conclusion

We have presented the modified algorithms for testing observability and accessibility for discrete-time polynomial systems based on those proposed in [6].

Our method also use Gröbner basis computation for decision problems and incorporates the strategy to improve efficiency in checking equivalence of ideals.

And we also have shown that we can extend our algorithm for forward accessibility to backward accessibility and transitivity by employing the well-known definition of invertibility of polynomial map and using Gröbner basis technique.

We should examine the actual efficiency of our method by experiments on computer. This is one of our important future works.

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